Field Extensions and Galois Theory
Mid-Semestral Exam
Sept. 2023

Time: 2 hours

Max score: 30

Answer all questions.

(1) Prove or disprove. (3×5)

(a) There exists an irreducible polynomial of degree 3 over $\mathbb{R}$.

(b) Let α denote the positive real fourth root of 2. Then there are two intermediate subfields of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

(c) Every finite extension is contained in a Galois extension.

(d) The polynomial $x^2 - t$ is irreducible and inseparable over $\mathbb{F}_2(t)$, the field of rational functions over $\mathbb{F}_2$.

(e) If L/F and K/L are Galois extensions, then K/F is also a Galois extension.

(2) (a) Define a constructible real number.

(b) Prove that if $a_1, \ldots, a_m$ are constructible real numbers, then there is a chain of subfields of $\mathbb{R}$:

$$\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n = K \subset \mathbb{R}$$

such that

(i) $a_i \in K$ for all $i = 1, \ldots, m$ and

(ii) F_{i+1} is a quadratic extension of F_i, for each $i = 0, \ldots, n-1$.

(c) Conversely, show that if L_i's are fields such that

$$\mathbb{Q} = L_0 \subset L_1 \subset \cdots \subset L_n = K \subset \mathbb{R},$$

and L_{i+1} is a quadratic extension of L_i for each $i = 0, \ldots, n-1$, then every element of K is constructible. $(1 + 3 + 1)$

(3) (a) Let $\Phi_n(x)$ denote the n-th cyclotomic polynomial. Prove that for n odd, $n > 1$,

$$\Phi_{2n}(x) = \Phi_n(-x).$$

(b) Let F be a field of characteristic p. If K is a finite extension of F such that $[K : F]$ is relatively prime to p, show that K is separable over F. $(2 + 3)$

(4) (a) Find the splitting field K in $\mathbb{C}$ of the polynomial $x^4 - 4x^2 - 1$ over $\mathbb{Q}$.

(b) Determine the Galois group of this splitting field over $\mathbb{Q}$.

(c) Describe the lattices of subfields and the subgroups of the Galois group. $(1 + 2 + 2)$

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